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Dense packings of congruent circles in a circle

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Abstract

The problem of finding packings of congruent circles in a circle, or, equivalently, of spreading points in a circle, is considered. Two packing algorithms are discussed, and the best packings found of up to 65 circles are presented.

1. Introduction

Problems of packing congruent circles in different geometrical shapes in the plane were raised in the 1960s, and many results — mainly for small packings — were obtained. Excellent sources of results and open problems are [3, 4]. The development of new, effective optimization algorithms for packing problems and the ever-increasing performance of computing systems have recently brought these problems into focus again; computer-aided methods can now be used to construct good large packings.

We consider the problem of packing congruent circles inside a larger circle which, without loss of generality, is assumed to be of unit radius. Given n, we want to place n congruent circles without overlaps inside the larger circle in such a way that their common radius is as large as possible. We denote the maximum attainable radius of the circles by r_n and we call the corresponding placement an *optimal* packing.

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Instead of fixing the radius of the larger circle and searching for the maximum radius of the circles in the packing, one can equivalently search for the minimum ratio of the radius of larger circle to the radius of the circles in the packing without fixing either one. The latter minimum is denoted by $D_n = 1/r_n$. One more parameter that can be optimized is the density of a packing, which is the area occupied by the circles of the packing divided by the area of the larger, enclosing circle.

This circle packing problem has another equivalent presentation, where n points (rather than circles) are placed inside a circle with unit radius. The goal is to maximize the minimum pairwise distance between the points. If d_n denotes this maximum, we have the following relations:

$$d_n = \frac{2r_n}{1-r_n}, \qquad D_n = 1 + \frac{2}{d_n}.$$

To generate packings we have implemented two different algorithms, which are both stochastic. These algorithms are discussed in Section 3. One of them uses ordinary non-linear optimization algorithms with an approximate cost function while the other simulates the idealized movement of billiard balls inside a circular container. The computer programs have been executed repeatedly for each value of n, and the best packings found in these runs have been chosen. In most cases, the independent algorithms found the same best packing; this increases our confidence in the quality of the packings to be presented.

In the next section, earlier results on packing circles in a circle are surveyed. Optimal packings of up to 11 circles and the best known packings of up to 20 circles are presented. Our two packing algorithms are discussed in detail in Section 3. The best packings of 21 to 65 circles found are presented and discussed in Section 4, and the concluding remarks are given in Section 5.

2. Earlier results

Kravitz [10] was, to our knowledge, the first to consider the problem of packing n congruent circles in a circle. In [10] packings of up to 19 circles are given without any optimality proofs.[3] Graham [6] and Pirl [17] independently proved optimality of packings of up to 7 and 10 circles, respectively. Pirl also presented good packings of up to 19 circles; some of these packings (for $n = 14, 16, 17$) were later improved by Goldberg [5], who also gave a packing of 20 circles. Goldberg's packing of 17 circles was further improved by Reis [18], who extended the range of n to 25. The packing of $n = 25$ is improved in this paper. Recently, Melissen [13] proved the optimality for the case $n = 11$.

[3] In Fig. 11 in [10] the value of R/r is erroneous; the last digit should be 6 instead of 4.

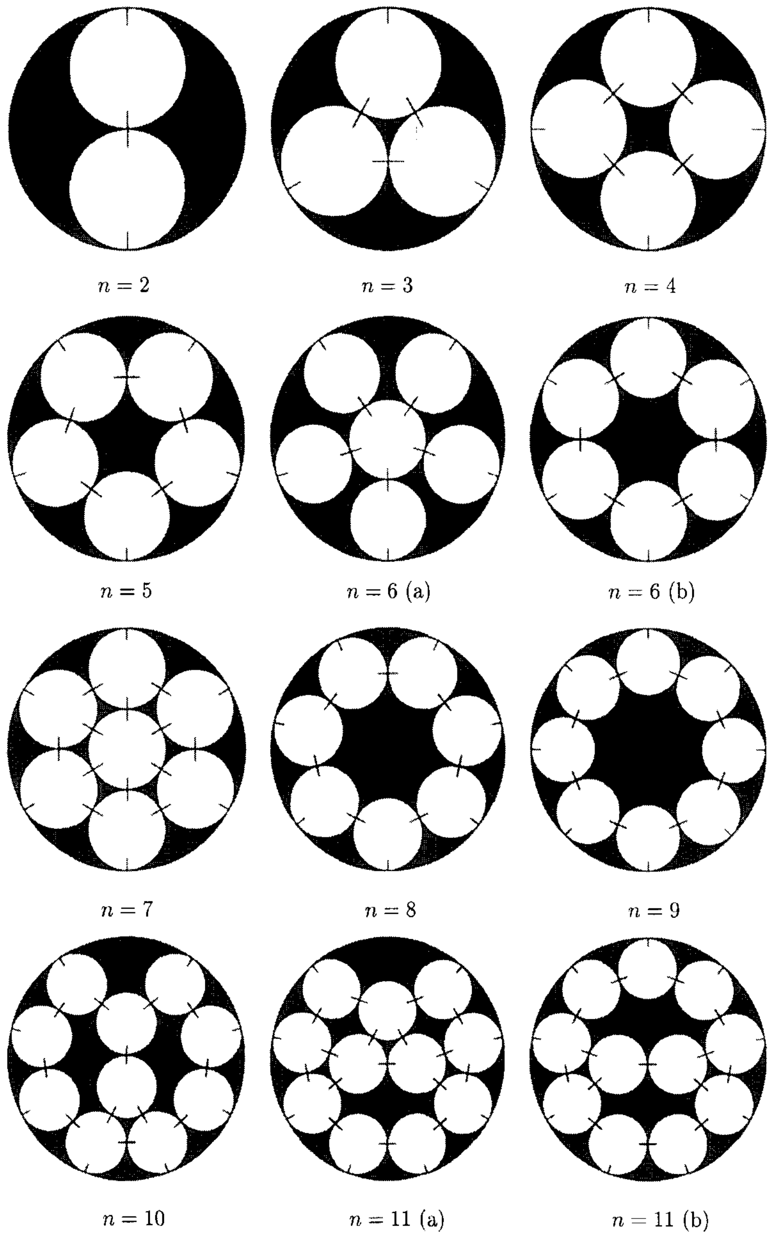

Fig. 1. Optimal packings of 2–11 circles in a circle.

In Fig. 1, the optimal packings of up to 11 circles are presented. The packings of 12 to 20 circles, conjectured to be optimal, are depicted in Fig. 2, excluding the case of 18 circles, which is displayed separately in Fig. 3.

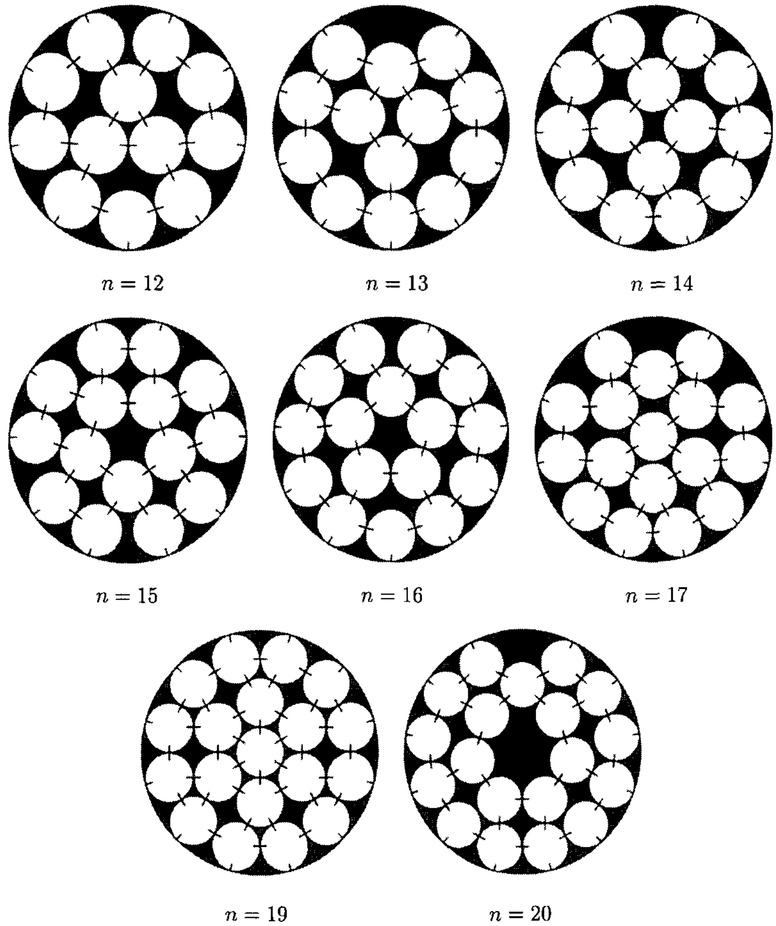

Fig. 2. Conjecturally optimal packings of 12–17 and 19–20 circles in a circle.

A peculiarity of the 18-circle case is that the best known packings of 18 circles have the same r as the best known packing of 19 circles. Three different, equally dense packings of 18 circles can be obtained by removing a circle in the packing of 19 circles in Fig. 2; see packings 18(a)–18(c) in Fig. 3. (A packing obtained by a congruence transformation, that is, by rotation or reflection, from another is considered the same.) In addition to these three packings, which apparently were the only ones known before, there are at least 7 more equally good packings. We suspect that there is no 11th equally good packing. At least, if one circle is removed from any of those 10 presumed best and then put back in the packing without overlaps with other circles, then one of these 10 packings is obtained. Furthermore, starting from any of these packings, all the others can be obtained with a series of such transformations.

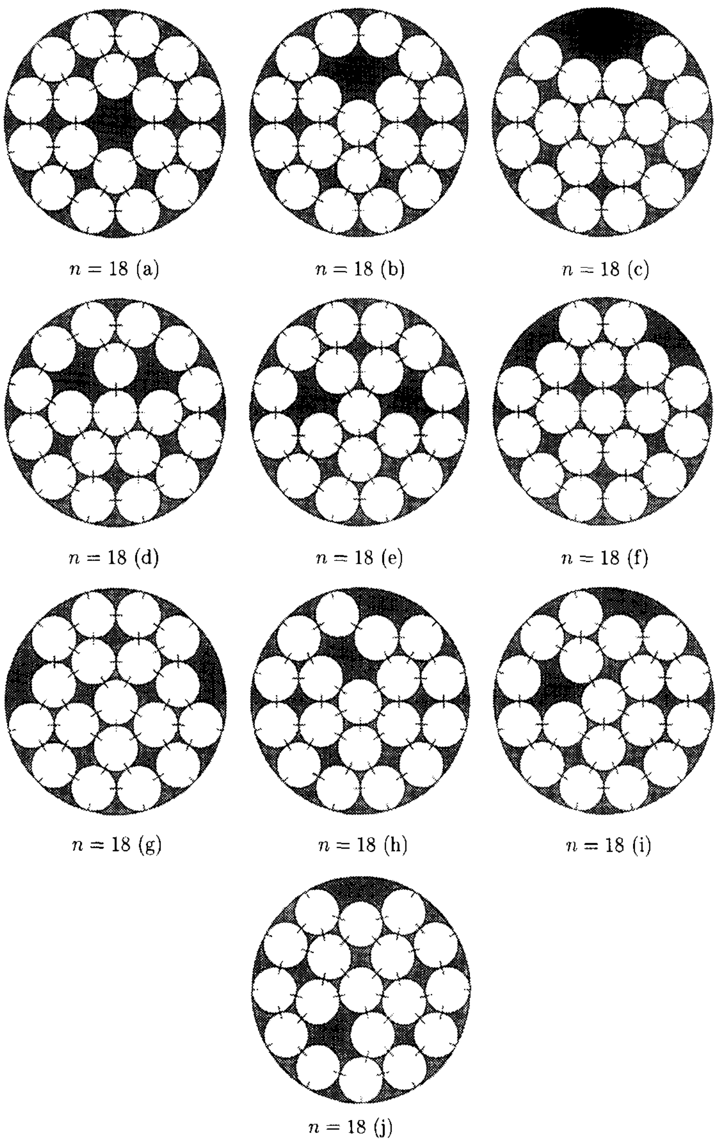

Fig. 3. Conjecturally optimal packings of 18 circles in a circle.

The case of 6 circles is analogous to that of 18 circles; different packings can be obtained from the 7-circle packing by removing and reordering circles. There are more than one optimal packings also for $n = 11$.

Not very much is known for $n > 20$; our intention in this study is to explore these cases. One family of good packings has already been reported in [12]. In the next sections we shall present the packing algorithms used to produce good packings and give the best packings found for 65 or fewer circles.

3. Packing algorithms used

In this section, two different methods to generate good packings are presented. In both methods, we try to maximize the minimum pairwise distance among n points spread in the unit circle centered in the origin. We denote this minimum distance by the objective function

$$d(S) = \min\{\| s_i - s_j \| : 1 \leqslant i < j \leqslant n\}, \tag{1}$$

where $S = \{s_1, s_2, \ldots, s_n\}$ is the set of points in the unit circle. Now the packing problem can be formulated as an optimization problem

$$\max : d(S), \tag{2}$$

whose global optima are the (globally) optimal packings. A packing found by one of these methods can be further improved by identifying the contacts (bonds) between the circles and solving numerically the corresponding system of equations.

3.1. Optimization by repulsion forces

The difficulty with (1) when trying to apply an efficient local optimization algorithm is that (1) is not smooth, and, furthermore, most of the first derivatives of the function are zero almost everywhere in the feasible region. To overcome these difficulties, we approximate the original problem (2) by minimizing the objective function

$$\sum_{1 \leqslant i < j \leqslant n} \left(\frac{\lambda}{\| s_i - s_j \|^2} \right)^m, \tag{3}$$

where λ is a suitable scaling factor (cf. [1]). This can be seen as a potential energy function when there are repulsion forces between the points. The parameter m controls the strength of the repulsion forces. The optima of (3) approximate those of (1) in the sense that as m tends to infinity, only the shortest distance between the points becomes significant.

The cost function (3) is smooth everywhere except where two points coincide. In practice, a situation where two points coincide is almost impossible to obtain, because such locations in the solution space are surrounded by very high potential barriers.

A packing is formed by first finding a local optimum of (3) using a relatively low value of m, for example, $m = 80$. Then the value of m is increased and the optimization step is repeated. If the value of m is only slightly increased, the local optima of consecutive approximations (3) are very close to each other, so this optimization step is usually relatively fast. There is a trade-off when selecting the initial value of m: a small m gives faster convergence to the first local optimum but then some good local optima of (1) may be missed.

In the beginning of each optimization stage we use a simple steepest-descent search with Goldstein–Armijo backtracking line search. When the gradient becomes small enough we switch to a modified Newton method to get a higher (in this case quadratic) convergence rate.

Instead of doing the calculations in Cartesian coordinates, we transform the coordinates to make the resulting optimization problem unconstrained. Let us denote the transformed coordinates of a point $\boldsymbol{x} = (x_1, x_2)$ by $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \tilde{x}_2)$. The transformation used is defined by

$$x_1 = \sin(\tilde{x}_1)\sin(\tilde{x}_2), \qquad x_2 = \cos(\tilde{x}_1)\sin(\tilde{x}_2).$$

Now we have an unconstrained optimization problem in the transformed variables. We could use a constrained optimization algorithm, but in order to make the constraints linear, some kind of a transformation is needed anyway.

3.2. *Billiards simulation*

In the second packing algorithm we use the hard disks or *billiards* model of computational physics. Consider a set S of n points spread in the unit circle. Let $d = d(S)$ be the minimum pairwise distance among the points, as defined in (1). If we draw circles of diameter d around each point in S, the circles do not overlap. Now imagine that we relax the configuration S and let the circles, which can also be thought of as hard disks, move chaotically obeying the restrictions of not overlapping and not escaping the larger circle of radius $1 + d/2$ (so that the centers of the circles do not escape the unit circle).

While the disks are moving, the configuration is visiting a subset in the configuration space over which the maximum of (1) is being sought. If $d = 0$ this subset is the entire configuration space. Now as disks move we gradually increase d. Then the subset in the configuration space which is being visited by the changing configuration is gradually shrinking. In the limit, the subset reduces to a single configuration (up to the equivalence transformation of rotation), and thus a packing is achieved as a steady state of the system. This configuration is a local maximum of the function (1).

The computer algorithm realizes the scenario outlined above. The effect of chaotic motion is obtained by letting the disks experience elastic collisions at impacts beginning with a randomly chosen initial configuration at $d = 0$. Round-off errors in the computations work as an additional randomization factor.

The computations are implemented in an efficient, event-driven fashion where the algorithm proceeds directly from one collision to another and does not waste computations on disks that are not involved in a collision. A more detailed description of the algorithm is given in [11].

3.3. Tightening the packings

A characteristic feature of both methods discussed above is their slowing down near an optimum; it becomes progressively more difficult to obtain higher precision of the positions of the circle centers as the configuration approaches a steady state. The precision is also limited by the precision of the software used (both algorithms were implemented using double precision). Furthermore, the local optima of (3) with finite m differ slightly from the corresponding optima of (1). Thus, both algorithms yield what can be qualified as 'loose' packings.

To tighten the packings, a complementary procedure was developed that works as follows. The first step of the procedure is detecting points of contact in a packing obtained using the methods described above. We identify circles that touch each other or the area boundary. To know which circles are touching, we select a small positive threshold value and compare it to the computed gap between the circumferences of each pair of circles. Selecting the threshold is usually easy, because in the gap distribution there is most often a clearly visible jump between, say, gaps smaller than 10^{-12} and gaps larger than 10^{-6} of the diameter of the circles.

In the second step of the procedure, given the conjectural contacts, a system of non-linear equations is formed. The equations state that the distances between centers of contacting circles are equal and that the centers of circles touching the area boundary are located on the periphery of the unit circle. Often there are more equations than unknowns in the system, especially when the packing is symmetric.

The final step of the procedure is to solve the system of equations formed in the second step. Several numerical methods can be used, we use a modified Newton–Raphson method. Alternatively, the system can be formulated as a nonlinear least-squares minimization problem, for which several algorithms exist in the literature. Because we have a very good initial 'loose' solution, fast convergence to a 'tight' solution usually takes place, if the system has a solution. In finding the 'tight' solution we use high computational resolution.

If the right contacts are not identified in the first step, the final step usually does not converge. In work on another packing problem [16], it turned out that difficulties can arise especially when there is a large, slightly disturbed hexagonal array of circles as a part of the packing. Among the cases considered in the present paper, only the case of 53 circles was difficult. The billiards algorithm, however, was finally able to give a solution with such a precision that the true contacts could be identified.

Previous research has revealed that there are good circle packings in an equilateral triangle and in a square with very narrow gaps between the circles or between a circle and the boundary [7, 14, 16]. This is why the calculations in the final step presented above were done with a very high computational resolution. In all the packings of this paper, the maximum difference between the distances of the centers of contacting circles is less than 10^{-98} and the maximum off-placement of the centers of the periphery circles is less than 10^{-100} when the centers are contained in the unit circle. It is thus extremely unlikely that any of the packing structures presented in this paper turns out not to exist.

With this precision, we are confident about the existence of a packing if the system of equations has a solution where no circles overlap. The solution might exist and yet, the packing might not be *rigid*; that is, there might exist a continuous movement of a subset of circles, different from the rotation of the entire assembly, such that no circles overlap or penetrate the boundary. In some cases both the existence and the rigidity problem could be solved using the interval arithmetic package INTBIS [9].

Numerical evidence for the rigidity of the packings in this paper was obtained as follows. We solved the system of equations several times with different initial 'loose' solutions which were obtained by rotating and slightly perturbing one 'loose' solution. Each time the algorithm converged to the same 'tight' solution (up to a rotation) with a very high precision. If the packing were not rigid, infinitely many different 'tight' solutions would have existed in arbitrarily small neighborhood of the first found 'tight' solution, and it would be highly improbable that the algorithm would converge every time to the same 'tight' solution (within the very high computational resolution that was used).

4. Packings found

The best known packings of 21–32, 33–44, 45–56, and 57–65 circles are shown in Figs. 4, 5, 6, 7, respectively. The packings of 21–24 circles are from [18], and the packings of 36–38, 60–62 circles are from [12].

With 28, 29, 42, 44, and 58 circles there exist many inferior packings that are close to the optimum and have slightly different density and bond structure. The packings of 19, 37, and 61 circles are examples of curved hexagonal packings defined in [12].

Some properties of the best known packings are listed in Table 1.

For completeness, we have included all packings from 2 to 65 circles in Table 1. In the table, n denotes the number of circles and d is the distance between the centers of the circles when the centers are contained in a circle of radius 1. The density of the packing is shown in the next column. The number of loose circles and the number of contacts are shown in the next two columns. Finally, the symmetry group of the packing is given. Loose circles are not considered when determining the symmetry group; in all cases these can be placed to fulfill the symmetry imposed by the other circles. In some cases there exist more symmetric but slightly less dense packings than those in Table 1.

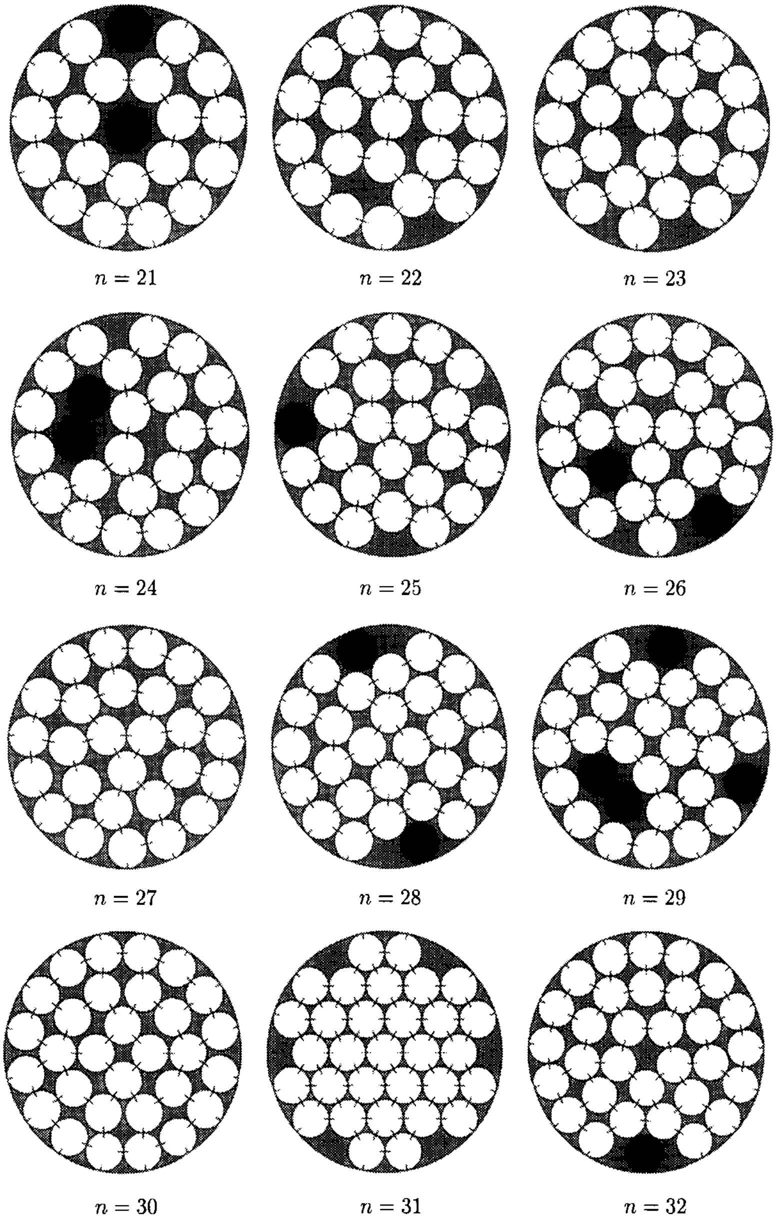

Fig. 4. Best known packings of 21–32 circles.

Note that in all cases, shown digits of d and *density* suffice to discern the best achieved packing from the next-best we got in the experiments. Some of the values of d in Table 1 can be expressed explicitly. For example, if there are k circles touching the larger circle and there are no gaps between these circles, then $d = 2\sin(\pi/k)$. This

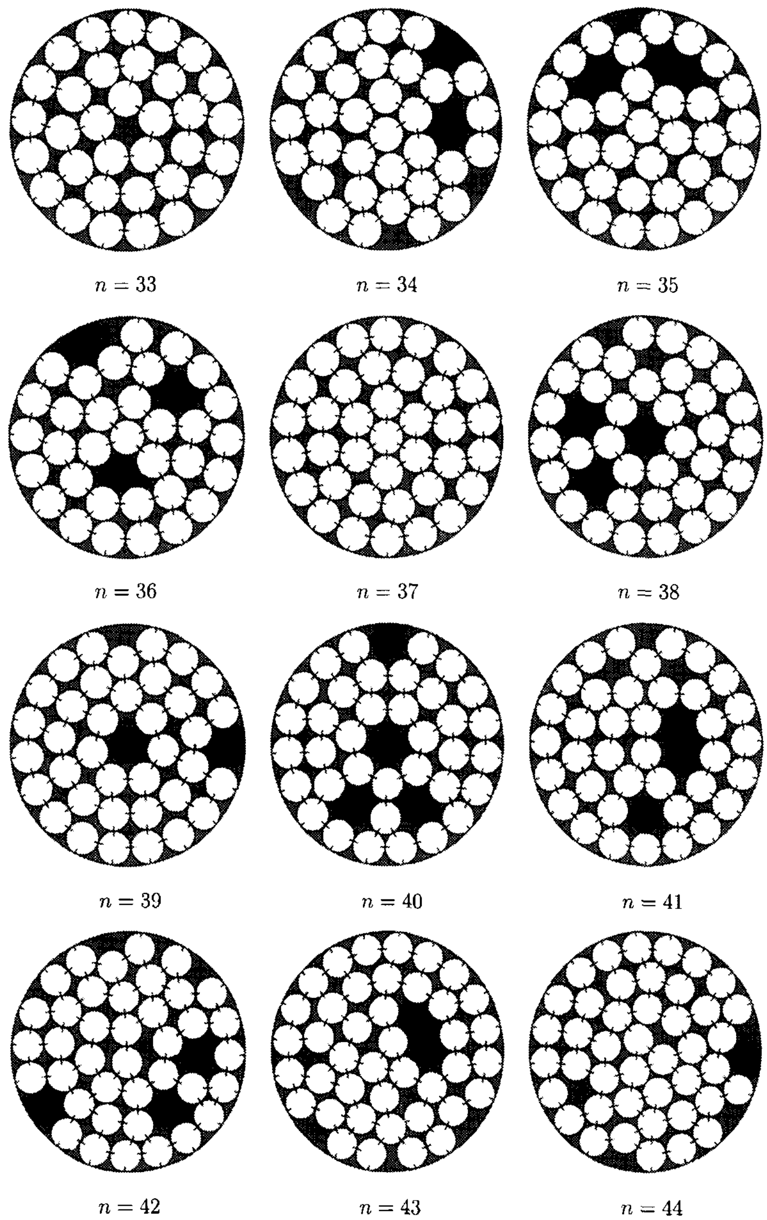

Fig. 5. Best known packings of 33–44 circles.

applies to the cases $n = 6$ and 7 with $k = 6$, $n = 8$ with $k = 7$, $n = 9$ with $k = 8$, $n = 11$ with $k = 9$, $n = 18$ and 19 with $k = 12$, $n = 37$ with $k = 18$, and $n = 61$ with $k = 24$. Furthermore, for $n = 31$ and 55, $d = \sqrt{1/7}$ and $\sqrt{1/13}$, respectively. In all cases, the exact value of d can be expressed as a solution of a system of equations.

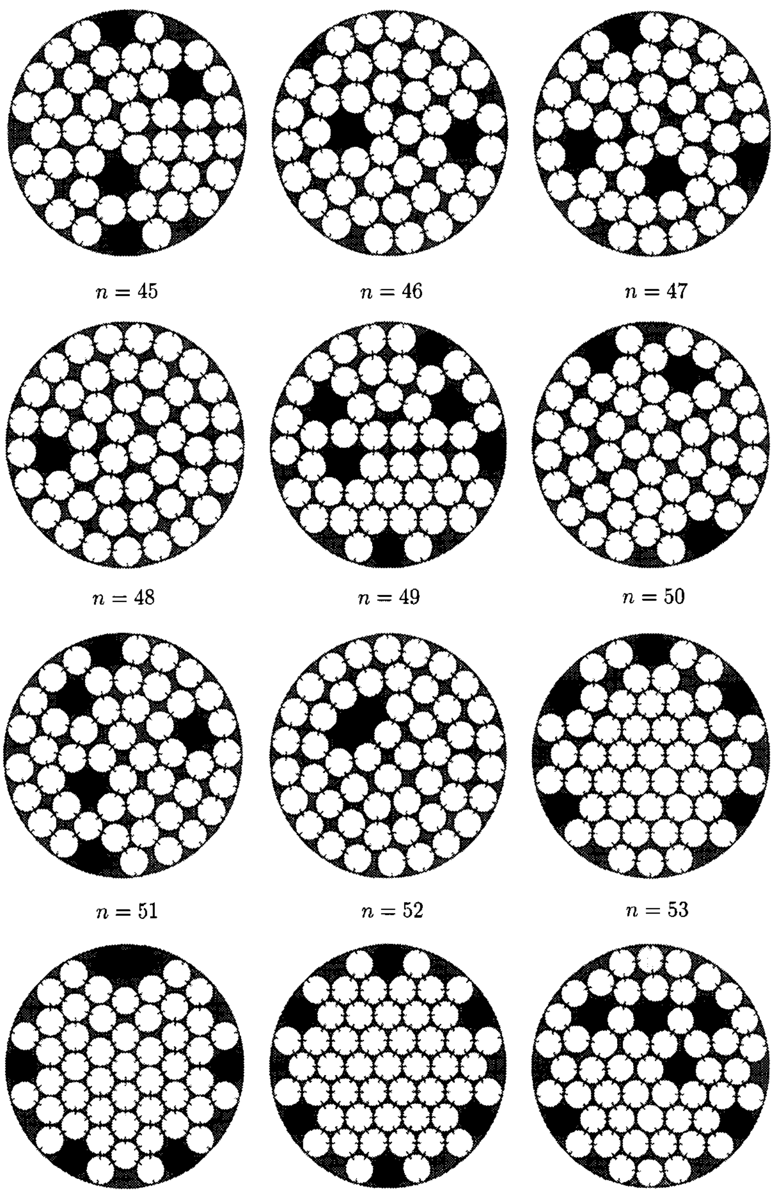

Fig. 6. Best known packings of 45–56 circles.

The packing 6(a) in Fig. 1 has one rigid circle surrounded by 5 loose circles that can be moved relatively to each other along the enclosing circle periphery. The number of contacts between the circles varies between 10 and 14 and the number of free circles is from 2 to 5. The symmetry group can be C_1, D_1, or D_5.

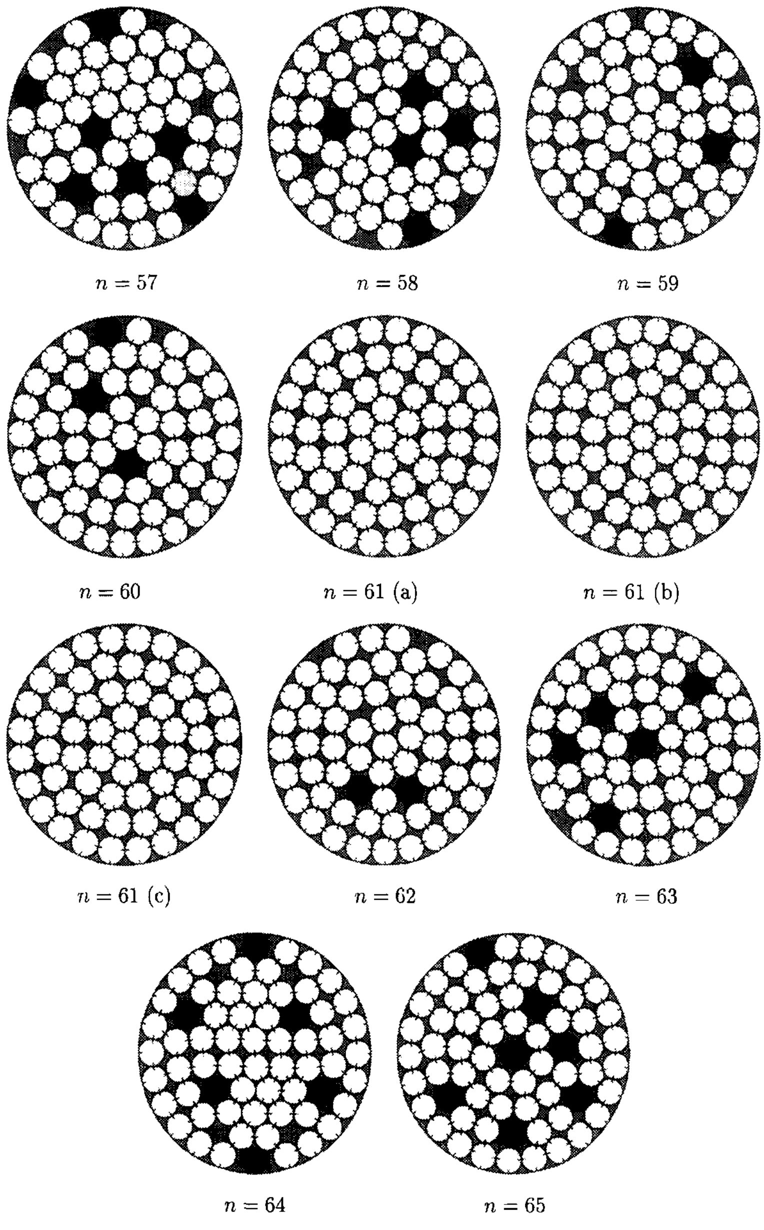

Fig. 7. Best known packings of 57–65 circles.

The packing of 25 circles improves on the best previously known packing for which $d = 0.420093512$ [18]. The new packing is almost symmetric and has one loose circle. The packing in [18] has two loose circles and is symmetric.

Table 1
Properties of the best packings

n	d	Density	Loose circles	Contacts	Symmetry group
2	2.000000000	0.500000000	0	3	D_2
3	1.732050808	0.646170928	0	6	D_3
4	1.414213562	0.686291501	0	8	D_4
5	1.175570505	0.685210244	0	10	D_5
6 (a)	1.000000000	0.666666667	2–5	10–14	See text
6 (b)	1.000000000	0.666666667	0	12	D_6
7	1.000000000	0.777777778	0	18	D_6
8	0.867767478	0.732502069	1	14	D_7
9	0.765366865	0.689407990	1	16	D_8
10	0.710978236	0.687797433	0	20	D_1
11 (a)	0.684040287	0.714460109	0	24	D_1
11 (b)	0.684040287	0.714460109	0	23	D_1
12	0.660152735	0.739021298	0	24	D_3
13	0.618033989	0.724465170	0	26	D_1
14	0.600884161	0.747252762	0	29	D_1
15	0.567962868	0.733759381	0	30	D_5
16	0.553185219	0.751097773	0	32	D_1
17	0.527421466	0.740302448	0	35	D_1
18 (a)	0.517638090	0.760918874	0	42	D_6
18 (b)	0.517638090	0.760918874	0	43	D_1
18 (c)	0.517638090	0.760918874	1	41	D_1
18 (d)	0.517638090	0.760918874	0	42	D_1
18 (e)	0.517638090	0.760918874	0	42	D_1
18 (f)	0.517638090	0.760918874	0	44	D_1
18 (g)	0.517638090	0.760918874	0	44	D_1
18 (h)	0.517638090	0.760918874	0	43	C_1
18 (i)	0.517638090	0.760918874	0	43	C_1
18 (j)	0.517638090	0.760918874	0	43	C_1
19	0.517638090	0.803192145	0	48	D_6
20	0.485163607	0.762248290	1	38	D_1
21	0.470331769	0.761232561	2	38	D_1
22	0.450478965	0.743480797	0	44	C_1
23	0.440024233	0.747984753	0	46	C_1
24	0.429953937	0.751378942	2	44	C_1
25	0.420802424	0.755401397	1	48	C_1
26	0.414235061	0.765434356	2	48	C_1
27	0.407631028	0.773959740	0	54	C_3
28	0.398808512	0.773919353	2	52	C_2
29	0.389211228	0.769590369	4	50	C_1
30	0.384782538	0.781006050	0	60	D_2
31	0.377964473	0.783164386	0	84	D_6

Table 1. (continued)

n	d	Density	Loose circles	Contacts	Symmetry group
32	0.368360556	0.774106260	1	62	C_1
33	0.364517627	0.784270529	0	66	C_1
34	0.356445428	0.777947412	2	66	C_1
35	0.351051420	0.780342492	2	66	C_1
36	0.348022566	0.790883977	3	66	C_1
37	0.347296355	0.809965138	0	90	C_6
38	0.335464260	0.784024549	3	70	C_1
39	0.330148274	0.782916696	2	74	C_1
40	0.326592122	0.788189950	4	72	C_1
41	0.319488189	0.777873793	3	76	C_1
42	0.315119609	0.778132009	3	78	C_1
43	0.311529156	0.781028893	2	82	C_1
44	0.307785274	0.782631916	1	86	C_1
45	0.304130893	0.784005834	4	82	C_1
46	0.300743743	0.785985500	2	88	C_1
47	0.297434477	0.787760508	4	86	C_1
48	0.294495666	0.790723359	1	94	C_1
49	0.290407648	0.787746731	6	87	C_1
50	0.287872703	0.791602691	3	94	C_1
51	0.284595948	0.791423233	5	92	C_1
52	0.282297789	0.795561354	2	100	C_1
53	0.278567684	0.792161024	5	101	C_1
54	0.277624221	0.802313910	6	103	D_1
55	0.277350098	0.815754986	6	126	D_6
56	0.270854069	0.796673375	6	100	C_1
57	0.268557864	0.798823587	7	101	C_1
58	0.265796493	0.798150306	5	107	C_1
59	0.263417848	0.799121719	3	112	C_1
60	0.261567159	0.802599163	3	115	C_1
61 (a)	0.261052384	0.813137360	0	144	C_6
61 (b)	0.261052384	0.813137360	0	144	C_6
61 (c)	0.261052384	0.813137360	0	144	C_6
62	0.255435495	0.795231105	2	120	C_1
63	0.253409898	0.796722906	5	116	C_1
64	0.251194079	0.796843244	6	116	C_1
65	0.249457513	0.799375705	7	116	C_1

5. Conclusions

The repulsion force optimization method in Section 3.1 requires fewer steps to obtain a packing configuration than the billiards method. On the other hand, these steps are computationally slower than those of the billiards algorithm. As a result, the time to reach a comparable vicinity of the optimum using the billiards method is roughly the same as that for the repulsion force method. However, if accurate localization of an optimum is required, our experiments have showed that the billiards algorithm performs better. As *global* optima are concerned, the ability for both methods to find them hinges on making many runs with different initial conditions.

In some cases we can make the packing algorithms faster by concentrating only on the packings having certain symmetries and thus reducing the number of variables [15]. However, while most of the smaller (optimal or conjecturally optimal) packings have symmetries, it seems that when the number of circles is increased, symmetric best known packings become more and more rare.

In [16] it was conjectured that given a packing of $n \geqslant 2$ circles in a square, there exists a packing of $n-1$ circles in the square with greater value of d. For packings in a circle a similar claim is not true; an optimal packing of 6 circles is obtained from the optimal packing of 7 circles by removing one circle. For packings in an equilateral triangle it has been conjectured that there are infinitely many cases where $n-1$ circles cannot be packed better than n circles, namely for $n = k(k+1)/2; k = 1,2,\ldots$ (the triangular numbers) [7]. Such infinite families are not known for packings in a circle.

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